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$Sp(n)U(1)$-connections with parallel totally skew-symmetric torsion

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Abstract

We consider the unique Hermitian connection with totally skew-symmetric torsion on a Hermitian manifold. We prove that if the torsion is parallel and the holonomy is $Sp(n)U(1) \subset U(2n) \times U(1)$, then the manifold is locally isomorphic to the twistor space of a quaternionic Kähler manifold with positive scalar curvature. If the manifold is complete, then it is globally isomorphic to such a twistor space.

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1. Introduction

Let (M^n, g) be an oriented Riemannian manifold. Let G be a subgroup of $SO(n)$ and P_G be a G-structure on M, i.e., P_G is a principal G-bundle which is subbundle of the bundle of oriented orthonormal frames $P_{SO(n)}$. Suppose that the Levi-Civita connection ∇ does not come from a connection on P_G. Which is the best connection on P_G in this case?

The first obvious choice is the canonical connection. It is the unique connection ∇^c whose torsion is the intrinsic torsion of the G-structure P_G. It can be thought of as the orthogonal projection of the Levi-Civita connection in the affine space of all G-connections in the following sense: $\nabla^c = \nabla + A^c$, where at each point $p \in M$, A^c_p is orthogonal to $(T_pM)^* \otimes \mathfrak{g}$ in $(T_pM)^* \otimes so(T_pM)$.

Another choice would be to replace the condition of vanishing torsion, which characterizes the Levi-Civita connection, by the requirement that the torsion is (covariantly) constant. This implies the existence of an invariant element of $(T_pM)^* \otimes so(T_pM)$ with respect to the holonomy group of the connection. If there is no such invariant element with respect to G itself, this would mean that the holonomy group is a proper subgroup of G, i.e., a further reduction of the structure group should be possible. Thus a G-connection with parallel torsion does not always exist.

A third possibility is a G-connection $\nabla^a = \nabla + A^a$, for which the potential A^a (or, equivalently, the torsion T^a) is totally skew-symmetric. The advantage of such a connection is that it has the same geodesics as the Levi-Civita

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connection. In particular, it is complete exactly when the metric is complete. In the general case though there is neither existence nor uniqueness of a G-connection with totally skew-symmetric torsion.

As an illustration let us consider the case of almost Hermitian structure, i.e., $G = U(n) \subset SO(2n)$. It is proved in [9] (see also [11]) that a $U(n)$-connection (or, in the established terminology, a Hermitian connection) ∇^a with totally skew-symmetric torsion exists iff the Nijenhuis tensor is totally skew-symmetric and in this case it is unique. The last condition means that the almost Hermitian manifold lies in the Gray–Hervella class $\mathcal{G}_1$, which contains the nearly Kähler and the Hermitian manifolds [13]. The connection ∇^a has been used by Bismut to prove a local index theorem for the Dolbeault operator on Hermitian non-Kähler manifolds [6]. The perfect situation occurs in the complementary case of nearly Kähler manifolds: the canonical connection has totally skew-symmetric torsion, which is furthermore parallel. The last result was proved by Kirichenko [16] (for another proof see [4]). On the other hand, on a manifold of class $\mathcal{G}_1$ which is not nearly Kähler (in particular, on a Hermitian manifold) ∇^a does not coincide with the canonical connection nor is its torsion T^a parallel in general.

In this paper we are interested in $Sp(n)U(1)$-structures, where $Sp(n)U(1)$ is considered as a subgroup of $U(2n) \times U(1) \subset U(2n+1) \subset SO(4n+2)$ by a certain inclusion ρ (see Section 2). More precisely, we study the $(4n+2)$-dimensional Hermitian manifolds whose unique Hermitian connection with totally skew-symmetric torsion ∇^a has holonomy contained in $\rho(Sp(n)U(1))$ and parallel torsion T^a.

In Section 3 we consider $(2m+2)$-dimensional Hermitian manifolds such that the holonomy group of ∇^a is a subgroup of $U(m) \times U(1)$ and the torsion T^a satisfies a simple algebraic condition (see Proposition 3.2). We show that there is an interesting correspondence between such manifolds and nearly Kähler manifolds with $Hol(\nabla^c) \subset U(m) \times U(1)$. This allows us to prove that the torsion T^a is parallel. Furthermore, we prove that if T^a is non-degenerate, then $m = 2n$ and $Hol(\nabla^a) \subset \rho(Sp(n)U(1))$. It follows also that the Ricci tensors of both ∇^a and the Levi-Civita connection ∇ are ∇^a-parallel and positive definite.

In Section 4 we consider the curvature tensor of a Hermitian manifold such that ∇^a has parallel torsion and holonomy $\rho(Sp(n)U(1))$. It turns out that it decomposes in a way very similar to the decomposition of the curvature of a quaternionic Kähler manifold. The methods of this section can be applied to give a proof of the above mentioned fact that the torsion of a nearly Kähler manifold is parallel. This is done using the first Bianchi identity and the representation theory of $U(n)$.

In Section 5 we show what Section 4 makes conceivable: there are examples on twistor spaces of quaternionic Kähler manifolds (we adopt the definition that a 4-dimensional manifold is quaternionic Kähler if it is self-dual and Einstein). The twistor space $\mathcal{Z}$ of a quaternionic Kähler manifold M' carries in a natural way two almost complex structures J_1 (integrable) and J_2 (non-integrable). They were first defined on twistor spaces of 4-dimensional manifolds in [3] and [8] respectively. On $\mathcal{Z}$ there exists also a one-parameter family of metrics h_t, $t > 0$, Hermitian with respect to both J_1 and J_2. If the base M' has positive scalar curvature, there are two particularly interesting values t_0 and t_1 of the parameter t, such that $(\mathcal{Z}, h_{t_0}, J_1)$ is Kähler and $(\mathcal{Z}, h_{t_1}, J_2)$ is nearly Kähler [10,23,19,2]. We prove that the connection ∇^a of $(\mathcal{Z}, h_{t_1}, J_1)$ has parallel torsion and holonomy $\rho(Sp(n)U(1))$.

In the next two sections we prove the main result of this paper: there are no other examples, either globally or locally.

The global result is contained in Theorem 6.1: a complete Hermitian manifold, such that ∇^a has parallel torsion and holonomy $\rho(Sp(n)U(1))$, is the twistor space $(\mathcal{Z}, h_{t_1}, J_1)$ of some compact quaternionic Kähler manifold M' with positive scalar curvature. In particular, there are only finitely many such manifolds in each dimension since the same is true for the compact quaternionic Kähler manifolds with positive scalar curvature [17,18]. In fact, the only known examples of compact quaternionic Kähler manifolds with positive scalar curvature are the Wolf spaces [24,5]. In this case $(\mathcal{Z}, h_{t_1}, J_1)$ is homogeneous naturally reductive. According to the results in [10,15,22,14] the Wolf spaces are the only compact quaternionic Kähler manifolds with positive scalar curvature in dimensions 4, 8 and 12. Therefore we get a complete list of the Hermitian manifolds of dimension 6, 10 and 14, satisfying the above conditions. In the proof of Theorem 6.1 we use the corresponding results about nearly Kähler manifolds of Belgun and Moroianu [4] and Nagy [20].

Section 7 is devoted to the proof of Theorem 7.1, which is the local version of Theorem 6.1. As a corollary we get a corresponding local result for nearly Kähler manifolds.

Finally, we should mention that in dimension 6 this paper covers one of the cases considered in [1]. The subject of [1] are the 6-dimensional Hermitian manifolds on which ∇^a has parallel torsion T^a. Thus T^a is invariant with respect to the holonomy group of ∇^a and this strongly restricts the possible holonomy groups. In [1] these possibilities are listed and the corresponding manifolds are studied.

2. Algebraic preliminaries

Let $T \cong \mathbb{R}^{2m+2}$ be the standard $(2m+2)$-dimensional real representation of $U(m+1)$ (in the following sections T will be the tangent space of a Hermitian manifold). Its complexification is $T^{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$, where $T^{1,0} \cong \mathbb{C}^{m+1}$ is the standard $(m+1)$-dimensional complex representation of $U(m+1)$ and $T^{0,1}$ is its conjugate. Denote by $\Lambda^{p,q}T^* \cong \Lambda^p(T^{1,0})^* \otimes \Lambda^q(T^{0,1})^*$ the space of (complex) (p,q)-forms on T. Let $\mathcal{H} \cong \mathbb{R}^{2m}$ (resp. $\mathcal{V} \cong \mathbb{R}^2$) be the standard $2m$-dimensional (resp. 2-dimensional) real representation of $U(m)$ (resp. $U(1)$), with further notation similar to the above for T. Then, as representations of $U(m) \times U(1)$,

$$T = \mathcal{H} \oplus \mathcal{V}, \quad T^{1,0} = \mathcal{H}^{1,0} \oplus \mathcal{V}^{1,0}, \quad T^{0,1} = \mathcal{H}^{0,1} \oplus \mathcal{V}^{0,1}.$$

Let us now consider the subgroup $Sp(n)U(1)$ of $U(2n)$ (as an abstract group it is isomorphic to $(Sp(n) \times U(1))/_{\mathbb{Z}_2}$). Define the inclusion $\rho : Sp(n)U(1) \longrightarrow U(2n) \times U(1) \subset U(2n+1)$ as follows: if $a \in Sp(n)$, $b \in U(1)$, then

$$\rho(ab) = \begin{pmatrix} ab & 0 \\ 0 & b^2 \end{pmatrix} \in U(2n) \times U(1) \subset U(2n+1).$$

If $n = 1$, $Sp(1)U(1) \cong U(2)$ and for $c \in U(2)$

$$\rho(c) = \begin{pmatrix} c & 0 \\ 0 & \det c \end{pmatrix} \in U(2) \times U(1) \subset U(3).$$

Denote by E the standard $2n$-dimensional complex representation of $Sp(n)$ and by $F(k)$ the complex (1-dimensional) representation of $U(1)$ with weight k. Since E is self-adjoint, by the definition of ρ we have that, as $\rho(Sp(n)U(1))$-representations,

$$\mathcal{H}^{1,0} \cong (\mathcal{H}^{0,1})^* \cong E \otimes F(1), \quad \mathcal{H}^{0,1} \cong (\mathcal{H}^{1,0})^* \cong E \otimes F(-1),$$
$$\mathcal{V}^{1,0} \cong (\mathcal{V}^{0,1})^* \cong F(2), \quad \mathcal{V}^{0,1} \cong (\mathcal{V}^{1,0})^* \cong F(-2).$$

Let $e_1, \ldots, e_{2n+1}$ be an orthonormal basis of $T^{1,0}$ such that $e_1, \ldots, e_{2n} \in \mathcal{H}^{1,0}$, $e_{2n+1} \in \mathcal{V}^{1,0}$ and

$$\omega_0 = \sum_{k=1}^{n} e^{2k-1} \wedge e^{2k} \in \Lambda^{2,0}\mathcal{H}^* \cong \Lambda^2 E \otimes F(-2)$$

is the 2-form corresponding to the $Sp(n)$-invariant 2-form on $\Lambda^2 E$ ($e^1, \ldots, e^{2n+1}$ is the basis dual to $e_1, \ldots, e_{2n+1}$).

Define

$$T_0 := \omega_0 \wedge \bar{e}^{2n+1} \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^* \subset \Lambda^{2,1}T^*.$$

T_0 is obviously $\rho(Sp(n)U(1))$-invariant and

$$|\lambda T_0 + \bar{\lambda}\overline{T}_0|^2 = 12n|\lambda|^2. \tag{2.1}$$

(Here and in the sequel we use the tensorial norms. The other popular convention is that the norm of a (skew-symmetric) k-form is its tensorial norm divided by $k!$.)

Proposition 2.1. *The subspace of $\Lambda^{2,1}T^*$ on which $\rho(Sp(n)U(1))$ acts trivially is 1-dimensional and is spanned by T_0.*

Proof. We have

$$\Lambda^{2,1}T^* \cong \Lambda^{2,0}T^* \otimes (T^{0,1})^* \cong (\Lambda^{2,0}\mathcal{H}^* \oplus (\mathcal{H}^{1,0})^* \otimes (\mathcal{V}^{1,0})^*) \otimes ((\mathcal{H}^{0,1})^* \oplus (\mathcal{V}^{0,1})^*).$$

Further, the decompositions of these tensor products into irreducible $\rho(Sp(n)U(1))$-representations are

$$\Lambda^{2,0}\mathcal{H}^* \otimes (\mathcal{H}^{0,1})^* \cong ((\mathbb{C} \oplus \Lambda_0^2 E) \otimes F(-2)) \otimes (E \otimes F(1)) \cong (E \oplus E \oplus \Lambda_0^3 E \oplus K) \otimes F(-1),$$

where K is the irreducible representation of $Sp(n)$ with highest weight $(2, 1, 0, \dots, 0)$,

$$\Lambda^{2,0}\mathcal{H}^* \otimes (\mathcal{V}^{0,1})^* \cong ((\mathbb{C} \oplus \Lambda_0^2 E) \otimes F(-2)) \otimes F(2) \cong \mathbb{C} \oplus \Lambda_0^2 E,$$
$$((\mathcal{H}^{1,0})^* \otimes (\mathcal{V}^{1,0})^*) \otimes (\mathcal{H}^{0,1})^* \cong (E \otimes F(-1) \otimes F(-2)) \otimes (E \otimes F(1)) \cong (\mathbb{C} \oplus \Lambda_0^2 E \oplus S^2 E) \otimes F(-2),$$
$$((\mathcal{H}^{1,0})^* \otimes (\mathcal{V}^{1,0})^*) \otimes (\mathcal{V}^{0,1})^* \cong (E \otimes F(-1) \otimes F(-2)) \otimes F(2) \cong E \otimes F(-1)$$

($\Lambda_0^2 E$ and K are 0 if $n = 1$ and the same is true for $\Lambda_0^3 E$ if $n \leq 2$).

Hence the only subspace of $\Lambda^{2,1}T^*$, on which $\rho(Sp(n)U(1))$ acts trivially, is contained in $\Lambda^{2,0}\mathcal{H}^* \otimes (\mathcal{V}^{0,1})^*$ and is obviously spanned by T_0. □

Corollary 2.2. *Consider the space of real $((2, 1) \oplus (1, 2))$-forms as a real $\rho(Sp(n)U(1))$-representation. Then the subspace on which $\rho(Sp(n)U(1))$ acts trivially has (real) dimension 2 and is spanned by $T_0 + \overline{T}_0$ and $iT_0 - i\overline{T}_0$.*

Proposition 2.3. *The subgroup of $U(2n + 1)$, which preserves T_0, is $\rho(Sp(n)U(1))$.*

Proof. We already know that T_0 is $\rho(Sp(n)U(1))$-invariant. Let $h \in U(2n + 1)$ be such that $h(T_0) = T_0$, i.e.,

$$h(\omega_0) \wedge h(\bar{e}^{2n+1}) = \omega_0 \wedge \bar{e}^{2n+1}. \tag{2.2}$$

Hence h preserves $(\mathcal{V}^{0,1})^* = span\{\bar{e}^{2n+1}\}$ since the splitting $T^{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$ is $U(2n + 1)$-invariant. Therefore

$$h = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in U(2n) \times U(1).$$

Let $a \in U(2n)$, $b \in U(1)$ be such that $c = ab$, $d = b^2$. Then $h(\bar{e}^{2n+1}) = b^2.\bar{e}^{2n+1}$, $h(\omega_0) = b^{-2}.a(\omega_0)$ and so $h(\omega_0) \wedge h(\bar{e}^{2n+1}) = a(\omega_0) \wedge \bar{e}^{2n+1}$. This and (2.2) imply $a(\omega_0) = \omega_0$, i.e., $a \in Sp(n)$ and therefore $h \in \rho(Sp(n)U(1))$. □

Let g denote the standard inner product on T and J be the standard complex structure (acting as multiplication by i on $T \cong \mathbb{C}^{2n+1}$). Define $K, I \in End(T)$ by

$$g((K + iI)X, Y) = 2\omega_0(X, Y), \quad X, Y \in T. \tag{2.3}$$

Then K and I vanish on $\mathcal{V}$, preserve $\mathcal{H}$, $K_{|\mathcal{H}}$ and $I_{|\mathcal{H}}$ are orthogonal with respect to $g_{|\mathcal{H}}$ and $I_{|\mathcal{H}}$, $J_{|\mathcal{H}}$, $K_{|\mathcal{H}}$ satisfy the quaternionic identities. Since $span_{\mathbb{C}}\{\omega_0\}$ is $\rho(Sp(n)U(1))$-invariant, $span\{K, I\}$ is $\rho(Sp(n)U(1))$-invariant. Thus $span\{I_{|\mathcal{H}}, J_{|\mathcal{H}}, K_{|\mathcal{H}}\}$ is a $\rho(Sp(n)U(1))$-invariant quaternionic structure on $\mathcal{H}$ compatible with $g_{|\mathcal{H}}$.

Later we shall need also a second inclusion $\rho_2 : Sp(n)U(1) \longrightarrow U(2n) \times U(1) \subset U(2n + 1)$, defined as follows: if $a \in Sp(n)$, $b \in U(1)$, then

$$\rho_2(ab) = \begin{pmatrix} ab & 0 \\ 0 & b^{-2} \end{pmatrix} \in U(2n) \times U(1)$$

$$\left(\text{if } n = 1, \text{ we have } \rho_2 : U(2) \longrightarrow U(2) \times U(1): \right.$$

$$\left. \rho_2(c) = \begin{pmatrix} c & 0 \\ 0 & (\det c)^{-1} \end{pmatrix}\right).$$

3. Hermitian manifolds with $Hol(\nabla^a) \subset U(m) \times U(1)$

Let (M, g, J) be an almost Hermitian manifold. We denote the Levi-Civita connection by ∇ and the Kähler form by Ω,

$$\Omega(X, Y) = g(JX, Y).$$

We shall assume that (M, g, J) belongs to the class $\mathcal{G}_1$ of Gray and Hervella [13]. This class is characterized by the property that the Nijenhuis tensor N is totally skew-symmetric. As shown in [9], this is equivalent to the existence of a Hermitian connection ∇^a with totally skew-symmetric torsion T^a and this connection is furthermore unique. It is given by

$$\nabla^a = \nabla + \frac{1}{2}T^a,$$
$$T^a = -d^c\Omega + N,$$

where $d^c\Omega(X, Y, Z) = -d\Omega(JX, JY, JZ)$. The class $\mathcal{G}_1$ contains as subclasses the Hermitian manifolds and the nearly Kähler manifolds. They can be distinguished in terms of the torsion T^a as follows: the manifold is Hermitian (i.e., J is integrable) iff T^a is a $((2, 1) \oplus (1, 2))$-form and in this case $T^a = -d^c\Omega$, and it is nearly Kähler iff T^a is a $((3, 0) \oplus (0, 3))$-form. In the latter case ∇^a coincides with the canonical Hermitian connection ∇^c.

Proposition 3.1. *Let (M, g, J) be Hermitian, $Hol(\nabla^a) = \rho(Sp(n)U(1))$ and $\nabla^a T^a = 0$. Then $T^a = \lambda T_0 + \bar{\lambda}\overline{T}_0$, where $\lambda \in \mathbb{C}$ is a constant and T_0 is the ∇^a-parallel tensor field defined by the $\rho(Sp(n)U(1))$-invariant tensor T_0 from Section 2.*

Proof. $\nabla^a T^a = 0$ implies that T^a is invariant with respect to $Hol(\nabla^a) = \rho(Sp(n)U(1))$ and the assertion follows from Corollary 2.2. □

Let (M^{2m+2}, g, J) belong to the class $\mathcal{G}_1$ and $Hol(\nabla^a) \subset U(m) \times U(1)$. Then we have a ∇^a-parallel, orthogonal and J-invariant splitting $TM = \mathcal{H} \oplus \mathcal{V}$, where $\dim_{\mathbb{R}} \mathcal{H} = 2m$, $\dim_{\mathbb{R}} \mathcal{V} = 2$. We can define an orthogonal with respect to g almost complex structure $\hat{J}$ by

$$\hat{J}_{|\mathcal{H}} = J_{|\mathcal{H}}, \quad \hat{J}_{|\mathcal{V}} = -J_{|\mathcal{V}}.$$

Proposition 3.2. *Let (M, g, J) be a Hermitian manifold with $Hol(\nabla^a) \subset U(m) \times U(1)$ and $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^*$. Then $(M, g, \hat{J})$ is nearly Kähler and its canonical connection coincides with ∇^a. Conversely, if (M, g, J) is nearly Kähler with $Hol(\nabla^a) \subset U(m) \times U(1)$ and $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^*$, then $(M, g, \hat{J})$ is Hermitian and the unique Hermitian connection with totally skew-symmetric torsion coincides with ∇^a.*

Remark. The condition $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^*$ is automatically satisfied for a 6-dimensional nearly Kähler manifold.

Proof. The definition of $\hat{J}$ implies that $\nabla^a \hat{J} = 0$. Hence ∇^a is the Hermitian connection with totally skew-symmetric torsion also for $(M, g, \hat{J})$.

If (M, g, J) is Hermitian and $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^*$, then T^a is a $((3, 0) \oplus (0, 3))$-form with respect to $\hat{J}$ and therefore $(M, g, \hat{J})$ is nearly Kähler.

If (M, g, J) is nearly Kähler with $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^*$, then T^a is a $((2, 1) \oplus (1, 2))$-form with respect to $\hat{J}$ and therefore $(M, g, \hat{J})$ is Hermitian. □

Since the torsion of the canonical connection of a nearly Kähler manifold is parallel [16], we get

Corollary 3.3. *If (M, g, J) is a Hermitian manifold with $Hol(\nabla^a) \subset U(m) \times U(1)$ and $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^*$, then $\nabla^a T^a = 0$.*

Remark. The condition $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^*$ implies that Ω is co-closed, i.e., (M, g, J) is semi-Kähler (or balanced).

We summarize now some simple facts about Riemannian manifolds (M, g) with metric connection $\nabla^a = \nabla + \frac{1}{2}T^a$ with totally skew-symmetric torsion T^a, such that $\nabla^a T^a = 0$. In this case the first Bianchi identity is

$$\mathfrak{S}_{X,Y,Z} R^a(X, Y, Z, W) = \sigma_{T^a}(X, Y, Z, W), \tag{3.4}$$

where $\mathfrak{S}_{X,Y,Z}$ denotes a cyclic sum with respect to X, Y, Z and $\sigma_{T^a} \in \Lambda^4 T^*M$ is defined by

$$\sigma_{T^a}(X, Y, Z, W) = \mathfrak{S}_{X,Y,Z}\, g(T^a(X, Y), T^a(Z, W)). \tag{3.5}$$

(For the curvature tensors we use the following convention:

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]},\qquad R(X,Y,Z,W)=g(R(X,Y)Z,W).)$$

The Bianchi identity (3.4) implies $R^a(X,Y,Z,W)=R^a(Z,W,X,Y)$.

Since $\nabla^a T^a=0$, we have $R^a(U,V)(T^a)=0$, i.e.,

$$\mathfrak{S}_{X,Y,Z}R^a(U,V,X,T^a(Y,Z))=0. \tag{3.6}$$

The explicit relation between R^a and the curvature R of the Levi-Civita connection is given by

$$\begin{aligned}R^a(X,Y,Z,W) = {} & R(X,Y,Z,W)+\frac{1}{2}g(T^a(X,Y),T^a(Z,W))\\ & +\frac{1}{4}g(T^a(Y,Z),T^a(X,W))-\frac{1}{4}g(T^a(X,Z),T^a(Y,W)).\end{aligned} \tag{3.7}$$

Therefore we get the following relations between the Ricci tensors and the scalar curvatures of ∇^a and ∇:

$$Ric^a=Ric-\frac{1}{4}r^a,\qquad s^a=s-\frac{|T^a|^2}{4}, \tag{3.8}$$

where $r^a(X,Y)=g(T^a(X,\cdot),T^a(Y,\cdot))$.

From now on we shall consider almost Hermitian manifolds (M,g,J) belonging to $\mathcal{G}_1$ such that T^a is non-degenerate, i.e., $T^a(X,\cdot)\neq 0$ for each $X\neq 0$. Notice that for Hermitian manifolds this condition is weaker than the requirement $\nabla_X J\neq 0$ for each $X\neq 0$. For nearly Kähler manifolds the two conditions are equivalent and the manifolds satisfying them are called *strict* nearly Kähler [12].

Proposition 3.4. *Let (M,g,J) be a Hermitian manifold such that $Hol(\nabla^a)\subset U(m)\times U(1)$, $T^a\in\Lambda^{2,0}\mathcal{H}^*\otimes\Lambda^{0,1}\mathcal{V}^*\oplus\Lambda^{0,2}\mathcal{H}^*\otimes\Lambda^{1,0}\mathcal{V}^*$ and T^a is non-degenerate. Then $m=2n$, $Hol(\nabla^a)\subset\rho(Sp(n)U(1))$ and $T^a=\lambda T_0+\bar\lambda\overline{T}_0$, where $\lambda\in\mathbb{C}\setminus\{0\}$ is a constant.*

Proof. From Corollary 3.3 we know that $\nabla^a T^a=0$. Since $\nabla^a J=0$, we have

$$R^a(JX,JY,Z,W)=R^a(X,Y,JZ,JW)=R^a(X,Y,Z,W),\quad Ric^a(JX,JY)=Ric^a(X,Y).$$

Let $\{e_\alpha\}$ be a basis of $T^{1,0}M$. Then the Bianchi identity (3.4) becomes

$$R^a{}_{\alpha\bar\beta\gamma\bar\delta}-R^a{}_{\gamma\bar\beta\alpha\bar\delta}=(\sigma_{T^a})_{\alpha\bar\beta\gamma\bar\delta}. \tag{3.9}$$

Contracting (3.6) with respect to V and X, we get

$$Ric^{a}{}_{A}^{S}T^a{}_{BCS}+g^{PQ}g^{RS}(R^a{}_{APBR}T^a{}_{CQS}+R^a{}_{APCR}T^a{}_{QBS})=0. \tag{3.10}$$

Let $A=\alpha$, $B=\bar\beta$, $C=\bar\gamma$. Since T^a is a $((2,1)\oplus(1,2))$-form, we obtain

$$Ric^{a}{}_{\alpha}^{\sigma}T^a{}_{\bar\beta\bar\gamma\sigma}+g^{\bar\delta\tau}g^{\epsilon\bar\sigma}(-R^a{}_{\alpha\bar\delta\bar\beta\epsilon}T^a{}_{\tau\bar\gamma\bar\sigma}+R^a{}_{\alpha\bar\delta\bar\gamma\epsilon}T^a{}_{\tau\bar\beta\bar\sigma})=0. \tag{3.11}$$

In our case $T^a=\omega\wedge\bar e^{m+1}+\bar\omega\wedge e^{m+1}$, where $\omega\in\Lambda^{2,0}\mathcal{H}^*$ and $e^{m+1}\in\Lambda^{1,0}\mathcal{V}^*$, $|e^{m+1}|=1$. Since T^a is non-degenerate, ω must be a non-degenerate 2-form on $\mathcal{H}^{1,0}$. Hence $\mathcal{H}^{1,0}$ is even dimensional, i.e., $m=2n$. We can take an orthonormal basis $e_1,\dots,e_{2n+1}$ of $T^{1,0}M$ so that $e_1,\dots,e_{2n}\in\mathcal{H}^{1,0}$, $e_{2n+1}\in\mathcal{V}^{1,0}$ and $\omega=\sum_{k=1}^n\lambda_k e^{2k-1}\wedge e^{2k}$, where $\lambda_k>0$, $k=1,\dots,n$.

Let $\alpha=2n+1$, $\beta=2k-1$, $\gamma=2k$ in (3.11). Then we get

$$\lambda_k Ric^{a}{}_{2n+1}^{2n+1}-\lambda_k R^a{}_{2n+1\overline{2n+1}2k-1\overline{2k-1}}-\lambda_k R^a{}_{2n+1\overline{2n+1}2k\overline{2k}}=0. \tag{3.12}$$

From (3.9)

$$R^a{}_{2n+1\overline{2n+1}2k-1\overline{2k-1}}=R^a{}_{2k-1\overline{2n+1}2n+1\overline{2k-1}}+(\sigma_{T^a})_{2n+1\overline{2n+1}2k-1\overline{2k-1}}.$$

The splitting $TM=\mathcal{H}\oplus\mathcal{V}$ is ∇^a-parallel and therefore is preserved by $R^a(X,Y)$. Hence $R^a{}_{2k-1\overline{2n+1}2n+1\overline{2k-1}}=0$. Thus

$$R^a{}_{2n+1\overline{2n+1}2k-1\overline{2k-1}}=(\sigma_{T^a})_{2n+1\overline{2n+1}2k-1\overline{2k-1}}={\lambda_k}^2.$$

Similarly, $R^a{}_{2n+1\overline{2n+1}2k\overline{2k}} = \lambda_k{}^2$. Hence (3.12) yields

$$Ric^a{}^{2n+1}_{2n+1} = 2\lambda_k{}^2, \quad k = 1, \ldots, n.$$

This means that $\lambda_1 = \cdots = \lambda_n =: \lambda > 0$ and

$$T^a = \lambda \sum_{k=1}^{n} e^{2k-1} \wedge e^{2k} \wedge \bar{e}^{2n+1} + \lambda \sum_{k=1}^{n} \bar{e}^{2k-1} \wedge \bar{e}^{2k} \wedge e^{2n+1}.$$

Now Proposition 2.3 implies that $Hol(\nabla^a)$ is conjugate to a subgroup of $\rho(Sp(n)U(1))$. □

The computations in the above proof yield the following.

Corollary 3.5. *Let (M, g, J) be as in Proposition 3.4. Then Ric^a, s^a, Ric, s are given by (4.16) and (4.19) below. In particular, Ric^a and Ric are positive definite and ∇^a-parallel and (M, g) is Einstein only if $n = 1$.*

Proof. From (3.10) with $A = \alpha$, $B = \bar{\beta}$, $C = \gamma$ and (3.11) in a similar way to in the proof of Proposition 3.4 we get

$$Ric^a{}^{\sigma}_{\alpha} = \begin{cases} (n+1)\lambda^2, & \alpha = \sigma < 2n+1, \\ 2\lambda^2, & \alpha = \sigma = 2n+1, \\ 0, & \alpha \neq \sigma. \end{cases}$$

This and (2.1) yield (4.16). (4.19) follows from (4.16) and (3.8). □

Remark. Let (M, g, J) be a strict nearly Kähler manifold. Then T^a is a $((3,0) \oplus (0,3))$-form. Taking $A = \alpha$, $B = \beta$, $C = \gamma$ in (3.10) we get

$$Ric^a{}^{\sigma}_{\alpha} T^a{}_{\beta\gamma\sigma} + g^{\bar{\delta}\tau} g^{\bar{\epsilon}\sigma} (R^a{}_{\alpha\bar{\delta}\beta\bar{\epsilon}} T^a{}_{\gamma\tau\sigma} - R^a{}_{\alpha\bar{\delta}\gamma\bar{\epsilon}} T^a{}_{\beta\tau\sigma}) = 0.$$

This together with (3.9) yields

$$Ric^a{}^{\sigma}_{\alpha} T^a{}_{\beta\gamma\sigma} + g^{\bar{\delta}\tau} g^{\bar{\epsilon}\sigma} ((\sigma_{T^a})_{\alpha\bar{\delta}\beta\bar{\epsilon}} T^a{}_{\gamma\tau\sigma} - (\sigma_{T^a})_{\alpha\bar{\delta}\gamma\bar{\epsilon}} T^a{}_{\beta\tau\sigma}) = 0,$$

which in global notation can be written as

$$T^a(Y, Z, Ric^a(X)) = \frac{1}{2} T^a(X, r^a(Y), Z) + \frac{1}{2} T^a(X, Y, r^a(Z)). \tag{3.13}$$

Hence $T^a(Y, Z, (\nabla^a_W Ric^a)(X)) = 0$ for each Y, Z. Since T^a is non-degenerate, this implies $(\nabla^a_W Ric^a)(X) = 0$. Thus $\nabla^a Ric^a = 0$ and by (3.8) also $\nabla^a Ric = 0$.

This proof is essentially due to Kirichenko [16] but he concludes wrongly from (3.13) that the manifold is Einstein. The correct formulation can be found in [20]. From (3.13) one can also see that Ric^a and r^a commute. Hence, there is an orthonormal basis $\{e_\alpha\}$ of $T^{1,0}M$ consisting of eigenvectors for both Ric^a and r^a. If the corresponding eigenvalues are μ_α and ν_α, then (3.13) shows that $\mu_\alpha = \frac{1}{2}(\nu_\beta + \nu_\gamma)$ whenever $T^a{}_{\alpha\beta\gamma} \neq 0$. Thus, since r^a is positive definite, we get that Ric^a (and by (3.8) also Ric) is positive definite. This has also been proved in [20].

From Propositions 3.2 and 3.4 we get

Corollary 3.6. *Let (M, g, J) be a strict nearly Kähler manifold with $Hol(\nabla^a) \subset U(m) \times U(1)$ and $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^*$. Then $m = 2n$ and $Hol(\nabla^a) \subset \rho_2(Sp(n)U(1))$.*

4. The curvature

In Section 3 we saw that if (M, g, J) is a Hermitian manifold with $Hol(\nabla^a) = \rho(Sp(n)U(1))$ and $\nabla^a T^a = 0$, then $T^a = \lambda T_0 + \bar{\lambda} \overline{T}_0$ and the curvature tensor R^a has the following properties:

$R^a(X, Y) \in \rho(sp(n) \oplus u(1))$,

R^a is symmetric with respect to the first and second pair of arguments,

$bR^a = \sigma_{T^a}$,

where $b : (T^*)^{\otimes 4} \longrightarrow (T^*)^{\otimes 4}$ is

$$(bR)(X, Y, Z, W) = \mathfrak{S}_{X,Y,Z} R(X, Y, Z, W).$$

Let $\mathfrak{R}^a$ denote the space of algebraic tensors with these properties, i.e.,

$$\mathfrak{R}^a = S^2(\rho(sp(n) \oplus u(1))) \cap b^{-1}(\mathbb{R}\sigma_{T_0+\overline{T}_0}).$$

Proposition 4.1. *The complexification of $\mathfrak{R}^a$ is isomorphic to $\mathbb{C} \oplus S^4E$ as a $\rho(Sp(n)U(1))$-representation.*

Proof. $\mathbb{R}\sigma_{T_0+\overline{T}_0}$ and $u(1) \cong \mathbb{R}$ are trivial representations of $\rho(Sp(n)U(1))$ and the complexification of $sp(n)$ is isomorphic to S^2E. Hence the complexification of $\mathfrak{R}^a$ is

$$\mathfrak{R}^a \otimes \mathbb{C} \cong S^2(S^2E \oplus \mathbb{C}) \cap b^{-1}(\mathbb{C}\sigma_{T_0+\overline{T}_0}).$$

We have

$$S^2(S^2E \oplus \mathbb{C}) \cong S^2(S^2E) \oplus S^2E \oplus \mathbb{C}$$

and

$$S^2(S^2E) \cong S^4E \oplus L \oplus \Lambda_0^2E \oplus \mathbb{C},$$

where L is the representation of $Sp(n)$ with highest weight $(2, 2, 0, \dots, 0)$ (if $n = 1$, L and Λ_0^2E are 0). Thus the decomposition of $S^2(S^2E \oplus \mathbb{C})$ into irreducible $\rho(Sp(n)U(1))$-representations is

$$S^2(S^2E \oplus \mathbb{C}) \cong S^4E \oplus L \oplus \Lambda_0^2E \oplus S^2E \oplus \mathbb{C} \oplus \mathbb{C}.$$

Taking particular representatives of these spaces and using Schur's Lemma one sees that $S^4E \subset \ker b$, that b is non-zero on L, Λ_0^2E, S^2E and therefore they are not contained in $\mathfrak{R}^a \otimes \mathbb{C}$, and that $b_{|\mathbb{C}\oplus\mathbb{C}}$ is injective and $b(\mathbb{C} \oplus \mathbb{C}) \supset \mathbb{C}\sigma_{T_0+\overline{T}_0}$. Thus $\mathfrak{R}^a \otimes \mathbb{C} \cong \mathbb{C} \oplus S^4E$. □

Corollary 4.2. *As a real representation of $\rho(Sp(n)U(1))$, $\mathfrak{R}^a \cong \mathbb{R} \oplus r(S^4E)$, where $r(S^4E)$ is the real representation underlying S^4E. If $R^a \in \mathfrak{R}^a$ and the corresponding torsion is $T^a = \lambda T_0 + \bar{\lambda}\overline{T}_0$, then with respect to the above isomorphism*

$$R^a = \frac{|T^a|^2}{48n} R_0^a + R_{\text{hyper}}, \tag{4.14}$$

where R_{hyper} has the properties of an algebraic hyper-Kähler curvature tensor on $\mathcal{H}$ and

$$\begin{aligned} R_0^a(X, Y, Z, W) = &\sum_{L \in \{\mathbf{1}, I, J, K\}} (g_{|\mathcal{H}}(LY, Z) g_{|\mathcal{H}}(LX, W) - g_{|\mathcal{H}}(LX, Z) g_{|\mathcal{H}}(LY, W)) \\ &- 2(g_{|\mathcal{H}}(JX, Y) g_{|\mathcal{H}}(JZ, W) + 2g_{|\mathcal{H}}(JX, Y) g_{|\mathcal{V}}(JZ, W) \\ &+ 2g_{|\mathcal{V}}(JX, Y) g_{|\mathcal{H}}(JZ, W) + 4g_{|\mathcal{V}}(JX, Y) g_{|\mathcal{V}}(JZ, W)). \end{aligned} \tag{4.15}$$

The Ricci tensor Ric^a and the scalar curvature s^a of R^a are

$$Ric^a = \frac{|T^a|^2}{12n}((n + 1)g_{|\mathcal{H}} + 2g_{|\mathcal{V}}), \qquad s^a = \frac{(n^2 + n + 1)|T^a|^2}{3n}. \tag{4.16}$$

Proof. One needs only to check that the preimage in $\mathbb{C} \oplus \mathbb{C}$ of σ_{T^a} with respect to b is $\frac{|T^a|^2}{48n} R_0^a$ (recall that $|T^a|^2 = 12n|\lambda|^2$). The formulae for Ric^a and s^a follow from the explicit form of R_0^a (or, alternatively, from Corollary 3.5). □

From (3.7) we obtain

Corollary 4.3. *As a $\rho(Sp(n)U(1))$-representation, the space $\mathfrak{R}$ of algebraic tensors with the properties of the Riemannian curvature tensor of a Hermitian manifold with $Hol(\nabla^a) = \rho(Sp(n)U(1))$ and $\nabla^a T^a = 0$ is $\mathfrak{R} \cong \mathbb{R} \oplus r(S^4E)$. With respect to this isomorphism, for $R \in \mathfrak{R}$, we have*

$$R = \frac{|T^a|^2}{48n} R_0 + R_{\text{hyper}}, \tag{4.17}$$

where R_{hyper} *has the properties of an algebraic hyper-Kähler curvature tensor on* $\mathcal{H}$ *and*

$$\begin{aligned}R_0(X,Y,Z,W) = {} & R^{g_{|\mathcal{H}}}_{\mathbb{H}P^n}(X,Y,Z,W)\\ & -\frac{1}{2}\sum_{L\in\{I,K\}}(g_{|\mathcal{H}}(LY,Z)g_{|\mathcal{H}}(LX,W)-g_{|\mathcal{H}}(LX,Z)g_{|\mathcal{H}}(LY,W)\\ & -2g_{|\mathcal{H}}(LX,Y)g_{|\mathcal{H}}(LZ,W))\\ & +\frac{1}{2}(g_{|\mathcal{H}}(Y,Z)g_{|\mathcal{V}}(X,W)+g_{|\mathcal{V}}(Y,Z)g_{|\mathcal{H}}(X,W)\\ & -g_{|\mathcal{H}}(X,Z)g_{|\mathcal{V}}(Y,W)-g_{|\mathcal{V}}(X,Z)g_{|\mathcal{H}}(Y,W))\\ & +\frac{3}{2}(g_{|\mathcal{H}}(JY,Z)g_{|\mathcal{V}}(JX,W)+g_{|\mathcal{V}}(JY,Z)g_{|\mathcal{H}}(JX,W)\\ & -g_{|\mathcal{H}}(JX,Z)g_{|\mathcal{V}}(JY,W)-g_{|\mathcal{V}}(JX,Z)g_{|\mathcal{H}}(JY,W)\\ & -2g_{|\mathcal{H}}(JX,Y)g_{|\mathcal{V}}(JZ,W)-2g_{|\mathcal{V}}(JX,Y)g_{|\mathcal{H}}(JZ,W))\\ & -8g_{|\mathcal{V}}(JX,Y)g_{|\mathcal{V}}(JZ,W)\end{aligned} \tag{4.18}$$

($R^{g_{|\mathcal{H}}}_{\mathbb{H}P^n}$ *is given by (5.21) with* g' *replaced by* $g_{|\mathcal{H}}$ *and* I', J', K' *replaced by* I, J, K*). The Ricci tensor Ric and the scalar curvature* s *of* R *are*

$$Ric=\frac{|T^a|^2}{24n}((2n+3)g_{|\mathcal{H}}+(n+4)g_{|\mathcal{V}}),\qquad s=\frac{(4n^2+7n+4)|T^a|^2}{12n}. \tag{4.19}$$

Proposition 3.4 yields

Corollary 4.4. *If* (M,g,J) *is as in Proposition 3.4, then* R^a *and* R *are as in Corollaries 4.2 and 4.3 and* $\nabla^a R_0^a=0$, $\nabla^a R_0=0$.

Remark. The results in this section have obvious counterparts for strict nearly Kähler manifolds with $Hol(\nabla^a)\subset U(m)\times U(1)$ and $T^a\in\Lambda^{2,0}\mathcal{H}^*\otimes\Lambda^{1,0}\mathcal{V}^*\oplus\Lambda^{0,2}\mathcal{H}^*\otimes\Lambda^{0,1}\mathcal{V}^*$ (and therefore with $Hol(\nabla^a)\subset\rho_2(Sp(n)U(1))$ by Corollary 3.6). One only needs to change the signs of the summands of the type $g_{|\mathcal{H}}(J\cdot,\cdot)g_{|\mathcal{V}}(J\cdot,\cdot)$ in the formulae for R_0^a and R_0.

Remark. The method of the proof of Proposition 4.1 can be used to prove the already mentioned fact that the canonical connection of a nearly Kähler manifold has parallel torsion. In general, the first Bianchi identity has the form

$$\mathfrak{S}_{X,Y,Z}R^a(X,Y,Z,W)=\mathfrak{S}_{X,Y,Z}(\nabla^a_X T^a(Y,Z,W)+g(T^a(X,Y),T^a(Z,W))).$$

Since $Hol(\nabla^a)\subset U(n)$, $R^a\in\Lambda^2T^*\otimes u(n)$. Furthermore, T^a is a $((3,0)\oplus(0,3))$-form and therefore $\nabla^aT^a\in T^*\otimes(\Lambda^{3,0}T^*\oplus\Lambda^{0,3}T^*)$. Let $\tau^a(X,Y,Z,W)=g(T^a(X,Y),T^a(Z,W))$. Then

$$\tau^a(X,Y,Z,W)=g((T^a)^{3,0}(X,Y),(T^a)^{0,3}(Z,W))+g((T^a)^{0,3}(X,Y),(T^a)^{3,0}(Z,W)).$$

Hence, complexifying all spaces, we obtain

$$b(R^a-\nabla^aT^a-\tau^a)=0$$

with $R^a\in\Lambda^2(T^{\mathbb{C}})^*\otimes u(n)^{\mathbb{C}}$, $\nabla^aT^a\in((T^{1,0})^*\oplus(T^{0,1})^*)\otimes(\Lambda^{3,0}T^*\oplus\Lambda^{0,3}T^*)$, $\tau^a\in\Lambda^{2,0}T^*\otimes\Lambda^{0,2}T^*$. Decomposing these spaces into irreducible $U(n)$-representations and taking particular representatives and using Schur's Lemma to determine the rank of b on the different components, we obtain

$$\begin{aligned}&((T^{1,0})^*\oplus(T^{0,1})^*)\otimes(\Lambda^{3,0}T^*\oplus\Lambda^{0,3}T^*)\\ &\quad=\Lambda^{4,0}T^*\oplus\Lambda^{0,4}T^*\oplus V(2,1,1,0,\dots,0)\oplus V(0,\dots,0,-1,-1,-2)\\ &\qquad\oplus\Lambda^{2,0}T^*\oplus\Lambda^{0,2}T^*\oplus V(1,1,1,0,\dots,0,-1)\oplus V(1,0,\dots,0,-1,-1,-1)\end{aligned} \tag{4.20}$$

($V(\alpha)$ is the irreducible representation of $U(n)$ with highest weight α) and

- b is injective on the first four components in (4.20) and they are not contained in $\Lambda^2(T^{\mathbb{C}})^* \otimes u(n)^{\mathbb{C}}$ and $\Lambda^{2,0}T^* \otimes \Lambda^{0,2}T^*$. Therefore $\nabla^a T^a$ has no components in them.
- $\Lambda^{2,0}T^*$ is contained twice in $\Lambda^2(T^{\mathbb{C}})^* \otimes u(n)^{\mathbb{C}}$, but $b_{|\Lambda^{2,0}T^*\oplus\Lambda^{2,0}T^*\oplus\Lambda^{2,0}T^*}$ is injective. Hence $\nabla^a T^a$ has no component in $\Lambda^{2,0}T^*$. In a similar way $\nabla^a T^a$ has no component in $\Lambda^{0,2}T^*$.
- $V(1,1,1,0,\dots,0,-1)$ is contained once in $\Lambda^2(T^{\mathbb{C}})^* \otimes u(n)^{\mathbb{C}}$, but $b_{|V(1,1,1,0,\dots,0,-1)\oplus V(1,1,1,0,\dots,0,-1)}$ is injective. Thus $\nabla^a T^a$ has no component in $V(1,1,1,0,\dots,0,-1)$. The same is true for $V(1,0,\dots,0,-1,-1,-1)$.

Hence $\nabla^a T^a = 0$.

5. Examples: The twistor spaces

Recall that a quaternionic Kähler manifold is a $4n$-dimensional Riemannian manifold (M', g') whose holonomy is contained in $Sp(n)Sp(1)$ if $n > 1$ or which is self-dual and Einstein if $n = 1$. If $n > 1$ an equivalent definition is to require the existence of a subbundle $Q' \subset End(TM')$ of rank 3 which is locally trivialized by three orthogonal almost complex structures I', J', K' satisfying the quaternionic identities. Such a bundle exists also if $n = 1$. In this case we choose $Q' = \Lambda^2_- M'$ (if we would like to choose the other possibility $Q' = \Lambda^2_+ M'$, we have to replace "self-dual" by "anti-self-dual" in the definition of quaternionic Kähler manifold).

Every quaternionic Kähler manifold is Einstein and its curvature has the form

$$R' = \frac{s'}{16n(n+2)} R_{\mathbb{H}P^n} + R'_{\mathrm{hyper}},$$

where s' is the (constant) scalar curvature, $R_{\mathbb{H}P^n}$ is the (parallel) curvature tensor of $\mathbb{H}P^n$,

$$\begin{aligned} R_{\mathbb{H}P^n}(X,Y,Z,W) &= g'(Y,Z)g'(X,W) - g'(X,Z)g'(Y,W) \\ &\quad + \sum_{L\in\{I',J',K'\}} (g'(LY,Z)g'(LX,W) - g'(LX,Z)g'(LY,W) - 2g'(LX,Y)g'(LZ,W)), \end{aligned} \tag{5.21}$$

and R'_{hyper} has the symmetries of a hyper-Kähler curvature tensor. If $n = 1$, $R_{\mathbb{H}P^n}$ is the curvature of S^4 with the metric with sectional curvature 4 and $R'_{\mathrm{hyper}} = W_+$ (the positive Weyl tensor).

The twistor space $\mathcal{Z}$ of a quaternionic Kähler manifold M' is the S^2-bundle over M' whose fibre at $p \in M'$ is $\mathcal{Z}_p = \{z \in Q'_p : z^2 = -\mathbf{1}\}$. A local trivialization $\psi = (\pi, \varphi)$ of $\mathcal{Z}$ is defined by a local frame I', J', K' of Q', which satisfies the quaternionic identities, as follows: if $z \in \mathcal{Z}_p$, $z = aI' + bJ' + cK'$, then $\varphi(z) = (a,b,c) \in S^2 \subset \mathbb{R}^3$.

The Levi-Civita connection defines a horizontal distribution $\mathcal{H}$ on $\mathcal{Z}$. Let $\mathcal{V}$ be the vertical distribution (tangent to the fibres). Two almost complex structures J_1 and J_2 and a one-parameter family of Riemannian metrics h_t, $t > 0$, are defined on $\mathcal{Z}$ in the following way: at $z \in \mathcal{Z}$

$J_{1|\mathcal{H}_z} = J_{2|\mathcal{H}_z}$ is the complex structure corresponding to z under the isomorphism of $\mathcal{H}_z$ and $T_{\pi(z)}M'$ given by the projection $\pi : \mathcal{Z} \longrightarrow M'$,

$J_{1|\mathcal{V}_z} = -J_{2|\mathcal{V}_z}$ corresponds to the standard complex structure of S^2 via ψ,

$h_{t|\mathcal{H}_z}$ corresponds to $g'_{\pi(z)}$ via π,

$h_{t|\mathcal{V}_z}$ corresponds via ψ to the metric with sectional curvature $\frac{1}{nt}$ on S^2,

$\mathcal{H}$ and $\mathcal{V}$ are orthogonal with respect to h_t.

Every two frames of Q' satisfying the quaternionic identities are related by an $SO(3)$-matrix and therefore the definition of J_1, J_2 and h_t is independent of the choice of ψ.

It is well known that J_1 is integrable and J_2 is not, that the Riemannian submersions $\pi : (\mathcal{Z}, h_t) \longrightarrow (M', g')$ have totally geodesic fibres and that J_1 and J_2 are orthogonal with respect to h_t.

The Hermitian structures (h_t, J_1) are semi-Kähler for each t (see [19,2]). If the scalar curvature s' of M' is positive, there exist two especially interesting values of the parameter t: $(\mathcal{Z}, h_t, J_1)$ is Kähler iff $t = t_0 := \frac{4(n+2)}{s'}$ and $(\mathcal{Z}, h_t, J_2)$ is nearly Kähler iff $t = t_1 := \frac{2(n+2)}{s'}$ (see [10,23,19,2]).

Let us consider $\nabla^{a,t}$, the Hermitian connection with totally skew-symmetric torsion of $(\mathcal{Z}, h_t, J_1)$. An immediate consequence of the results in [19,2] is that its torsion $T^{a,t}$ is given by

$$T^{a,t} = \frac{1}{\sqrt{2nt}}\left(2 - \frac{s't}{2(n+2)}\right)(\omega \wedge \bar{\alpha} + \bar{\omega} \wedge \alpha), \tag{5.22}$$

where $\omega \in \Lambda^{2,0}\mathcal{H}^*$ and $\alpha \in \Lambda^{1,0}\mathcal{V}^*$ are defined as follows:
Using the trivialization ψ we can consider the vertical vectors at $z \in \mathcal{Z}$ as elements of $Q'_{\pi(z)}$ by identifying $(a, b, c) \in \mathbb{R}^3$ with $aI' + bJ' + cK' \in Q'_{\pi(z)}$. Thus, if $U \in \mathcal{V}_z$, we can define a 2-form Ω_U on $T_{\pi(z)}M'$ by $\Omega_U(X, Y) = g'(U(X), Y)$. We fix $U \in \mathcal{V}^{1,0}$ with $|U| = 1$ and take $\alpha \in \Lambda^{1,0}\mathcal{V}^*$ to be the dual form of U and $\omega = \frac{\sqrt{2nt}}{2}\pi^*\Omega_U$.

We have

$$|T^{a,t}|^2 = \frac{6}{t}\left(2 - \frac{s't}{2(n+2)}\right)^2. \tag{5.23}$$

Now, using [19,2], one can prove that $\nabla^{a,t}T^{a,t} = 0$ iff $s' > 0$ and $t = t_0$ (in this case $T^{a,t_0} = 0$) or $t = t_1$. Furthermore, the splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ is ∇^{a,t_1}-parallel (in fact, the splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ is always parallel with respect to the canonical Hermitian connection of (h_t, J_2), never parallel with respect to the canonical Hermitian connection of (h_t, J_1) and parallel with respect to $\nabla^{a,t}$ only if $s' > 0$ and $t = t_1$). Thus, by Proposition 2.3 (or Proposition 3.4) $Hol(\nabla^{a,t_1}) \subset \rho(Sp(n)U(1))$.

Proposition 5.1. *Let (M', g') be a quaternionic Kähler manifold with positive scalar curvature. Then the connection ∇^{a,t_1} on the twistor space $(\mathcal{Z}, h_{t_1}, J_1)$ has parallel torsion and $Hol(\nabla^{a,t_1}) \subset \rho(Sp(n)U(1))$. If (M', g') is not locally symmetric, then $Hol(\nabla^{a,t_1}) = \rho(Sp(n)U(1))$.*

Proof. Only the last assertion remains to be proved.

Since $Hol(\nabla^{a,t_1}) \subset \rho(Sp(n)U(1))$, we have a ∇^{a,t_1}-parallel quaternionic structure on $\mathcal{H}$, defined as in Section 2. On the other hand, since $\pi_{*|\mathcal{H}_z} : \mathcal{H}_z \longrightarrow T_{\pi(z)}M'$ is an isomorphism, the quaternionic structure on M' also defines a quaternionic structure on $\mathcal{H}$. We are going to show that these two quaternionic structures coincide.

Fix $z \in \mathcal{Z}$ and choose the trivialization I', J', K' of Q' so that $J'_{\pi(z)} = z$. Then the vector $U \in \mathcal{V}_z^{1,0}$ in the definition of α and ω above can be taken to be $U = \frac{1}{\sqrt{2nt_1}}(K' + iI')$ and therefore $\omega = \frac{1}{2}\pi^*\Omega_{K'+iI'}$. Thus the quaternionic structure $span\{I_{|\mathcal{H}_z}, J_{|\mathcal{H}_z}, K_{|\mathcal{H}_z}\}$ on $\mathcal{H}_z$, defined by $Hol(\nabla^{a,t_1})$, is exactly the pull-back $span\{(\pi_{*|\mathcal{H}_z})^{-1}(I'), (\pi_{*|\mathcal{H}_z})^{-1}(J'), (\pi_{*|\mathcal{H}_z})^{-1}(K')\}$ of the quaternionic structure Q' on M'.

In order to simplify the expressions, from now on we write g instead of h_{t_1}, J instead of J_1, ∇^a instead of ∇^{a,t_1} and T^a instead of T^{a,t_1}.

Since $t_1 = \frac{2(n+2)}{s'}$, (5.22) and (5.23) become

$$T^a = \frac{1}{\sqrt{2nt_1}}(\omega \wedge \bar{\alpha} + \bar{\omega} \wedge \alpha), \tag{5.24}$$

$$|T^a|^2 = \frac{3s'}{n+2}. \tag{5.25}$$

Taking into account (5.25) and that the above defined quaternionic structures on $\mathcal{H}$ coincide, the explicit formulae for the curvature tensor R of the Levi-Civita connection of $(\mathcal{Z}, h_t)$ in [7,2], applied for $t = t_1$, show that R is given by (4.17), where R_{hyper} is the horizontal lift of R'_{hyper} (and also $R^{g|\mathcal{H}}_{\mathbb{H}P^n}$ is the horizontal lift of $R_{\mathbb{H}P^n}$). Now (5.24) and (3.7) imply that the curvature of ∇^a is given by (4.14), R_{hyper} being the same as above.

The Lie algebra $hol(\nabla^a)$ of $Hol(\nabla^a)$ is contained in $\rho(sp(n) \oplus u(1))$. On the other hand, $hol(\nabla^a)$ contains the algebra generated by $\{(\nabla^a)^k_{X_1,\ldots,X_k} R^a(X, Y) : k \geq 0\}$ (in fact, ∇^a is real analytic and therefore $hol(\nabla^a)$ is equal to this algebra but we do not need the real analyticity here).

From the definition of ρ we see that $\rho(u(1))$ is spanned by $J_{|\mathcal{H}} + 2J_{|\mathcal{V}}$. If $e_1, \ldots, e_{4n+2}$ is an orthonormal frame of $T\mathcal{Z}$, then (4.14) and (4.15) give

$$J_{|\mathcal{H}} + 2J_{|\mathcal{V}} = -\frac{12n}{(2n+1)|T^a|^2} \sum_{k=1}^{4n+2} R^a(e_k, Je_k).$$

Hence $\rho(u(1)) \subset hol(\nabla^a)$. Therefore to prove that $\rho(sp(n)) \subset hol(\nabla^a)$ it will be enough to show that the algebra generated by

$$A = \left\{\text{the } \rho(sp(n))\text{-part of } R^a(X^h, Y^h), (\nabla^a)^k_{X_1^h,\ldots,X_k^h} R^a(X^h, Y^h) : k \geq 1\right\}$$

contains $\rho(sp(n))$.

The $\rho(sp(n))$-part of $R^a(X^h, Y^h)$ is

$$\frac{|T^a|^2}{48n} \sum_{L \in \{\mathbf{1}, I, J, K\}} (g_{|\mathcal{H}}(LY^h, \cdot)LX^h - g_{|\mathcal{H}}(LX^h, \cdot)LY^h) + R_{\text{hyper}}(X^h, Y^h)$$

and, because of (5.25), this projects on

$$\frac{s'}{16n(n+2)} \sum_{L \in \{\mathbf{1}, I', J', K'\}} (g'(LY, \cdot)LX - g'(LX, \cdot)LY) + R'_{\text{hyper}}(X, Y),$$

which is exactly the $sp(n)$-part of $R'(X, Y)$.

Since π is a Riemannian submersion, ∇ projects on the Levi-Civita connection ∇' of g', i.e., $h\nabla_{X^h} Y^h = (\nabla'_X Y)^h$. T^a has no components in $\Lambda^3\mathcal{H}^*$ and therefore $h\nabla^a_{X^h} Y^h = h\nabla_{X^h} Y^h$. Thus ∇^a also projects on ∇' and since R_{hyper} projects on R'_{hyper} and the splitting $TM = \mathcal{V} \oplus \mathcal{H}$ is parallel with respect to ∇^a, $(\nabla^a)^k R^a = (\nabla^a)^k R_{\text{hyper}}$ for $k \geq 1$ projects on $\nabla'^k R'_{\text{hyper}}$.

Hence A projects on

$$B = \left\{\text{the } sp(n)\text{-part of } R'(X, Y), \nabla'^k_{X_1,\ldots,X_k} R'_{\text{hyper}}(X, Y) : k \geq 1\right\}.$$

(M', g') is quaternionic Kähler with non-zero scalar curvature which is not locally symmetric. Therefore $Hol(\nabla') = Sp(n)Sp(1)$. Since every quaternionic Kähler manifold is real analytic, $hol(\nabla') = sp(n) \oplus sp(1)$ is generated by $\{\nabla'^k_{X_1,\ldots,X_k} R'(X, Y) : k \geq 0\}$. But for $k \geq 1$ we have $\nabla'^k_{X_1,\ldots,X_k} R'(X, Y) = \nabla'^k_{X_1,\ldots,X_k} R'_{\text{hyper}}(X, Y) \in sp(n)$ and this implies that $sp(n)$ is generated by B. Hence $\rho(sp(n))$ is contained in the algebra generated by A. Thus $\rho(sp(n)) \subset hol(\nabla^a)$ and therefore $hol(\nabla^a) = \rho(sp(n) \oplus u(1))$, i.e., $Hol(\nabla^a) = \rho(Sp(n)U(1))$. □

Corollary 5.2. *Let (M', g') be a quaternionic Kähler manifold with positive scalar curvature. Then the canonical connection ∇^{a,t_1} of the nearly Kähler manifold $(\mathcal{Z}, h_{t_1}, J_2)$ has $Hol(\nabla^{a,t_1}) \subset \rho_2(Sp(n)U(1))$. If (M', g') is not locally symmetric, then $Hol(\nabla^{a,t_1}) = \rho_2(Sp(n)U(1))$.*

Remark. A locally symmetric quaternionic Kähler manifold M' has holonomy group $HSp(1)$ where $H \subset Sp(n)$ (see [5] for the list of possible groups H). The proof of Proposition 5.1 shows that in this case the connection ∇^{a,t_1} on the twistor space $(\mathcal{Z}, h_{t_1}, J_1)$ has $Hol(\nabla^{a,t_1}) = \rho(HU(1))$. In particular, for $M' = \mathbb{H}P^n$ again $Hol(\nabla^{a,t_1}) = \rho(Sp(n)U(1))$. A similar remark is true for $(\mathcal{Z}, h_{t_1}, J_2)$ (replace ρ by ρ_2).

6. The compact case

Theorem 6.1. *Let (M, g, J) be a complete Hermitian manifold such that $Hol(\nabla^a) \subset U(m) \times U(1)$, $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^*$ and T^a is non-degenerate. Then $m = 2n$ and (M, g, J) is isomorphic to the twistor space $(\mathcal{Z}, h_{t_1}, J_1)$ of some compact quaternionic Kähler manifold with positive scalar curvature.*

Proof. By Proposition 3.2 $(M, g, \hat{J})$ is a complete strict nearly Kähler manifold with $Hol(\nabla^a) \subset U(m) \times U(1)$ and $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^*$. It follows from the results in [20] (or [4] if $n = 1$) that $(M, g, \hat{J})$ is isomorphic to the twistor space $(\mathcal{Z}, h_{t_1}, J_2)$ of some compact quaternionic Kähler manifold with positive scalar curvature. Now the definitions of $\hat{J}$ and J_2 show that (M, g, J) is isomorphic to $(\mathcal{Z}, h_{t_1}, J_1)$. □

From the results of the previous sections we obtain

Corollary 6.2. *A manifold which satisfies the assumptions of Theorem 6.1 is compact, simply connected, has positive and ∇^a-parallel Ricci tensor, $\nabla^a T^a = 0$ and $Hol(\nabla^a) \subset \rho(Sp(n)U(1))$. If furthermore the quaternionic Kähler base is not a symmetric space of rank greater than one, then $Hol(\nabla^a) = \rho(Sp(n)U(1))$.*

Theorem 6.1 has also the following consequences:

- In each dimension the manifolds satisfying its conditions are finitely many since the same is true for the compact quaternionic Kähler manifolds with positive scalar curvature [17,18].
- The only known examples of compact quaternionic Kähler manifolds with positive scalar curvature are the Wolf spaces [24,5], which are symmetric. Their twistor spaces $(\mathcal{Z}, h_{t_1}, J_1)$ are homogeneous. In fact, $(\mathcal{Z}, h_{t_1})$ is a naturally reductive homogeneous space with canonical connection ∇^{a,t_1} iff the base manifold is symmetric (this follows from the proof of Proposition 5.1 for example).
- It is known that in dimensions 4, 8 and 12 there are no compact quaternionic Kähler manifolds with positive scalar curvature other than the Wolf spaces [10,15,22,14]. Hence the only manifolds of dimension 6, 10 and 14 which satisfy the conditions of Theorem 6.1 are the twistor spaces of $S^4 \cong \mathbb{H}P^1$, $\mathbb{C}P^2$; $\mathbb{H}P^2$, $Gr_2(\mathbb{C}^4)$, $G_2/SO(4)$; $\mathbb{H}P^3$, $Gr_2(\mathbb{C}^5)$, $\tilde{Gr}_4(\mathbb{R}^7)$.

7. The local case

The goal of this section is to prove the local version of Theorem 6.1.

Theorem 7.1. *Let (M, g, J) be a Hermitian manifold such that $Hol(\nabla^a) \subset U(m) \times U(1)$, $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^*$ and T^a is non-degenerate. Then $m = 2n$ and locally (M, g, J) is isomorphic to the twistor space $(\mathcal{Z}, h_{t_1}, J_1)$ of some $4n$-dimensional quaternionic Kähler manifold M' with positive scalar curvature. In particular, $\nabla^a T^a = 0$ and $Hol(\nabla^a) \subset \rho(Sp(n)U(1))$, with equality if M' is not locally symmetric of rank greater than one.*

We begin the proof with the following straightforward

Lemma 7.2. *Let $U, V \in \Gamma(\mathcal{V})$ and $X, Y \in \Gamma(\mathcal{H})$. Then*

$$\nabla_U V = {\nabla^a}_U V \in \mathcal{V},$$
$$\nabla_U X = {\nabla^a}_U X - \frac{1}{2}T^a(U, X) \in \mathcal{H},$$
$$h\nabla_X U = \frac{1}{2}T^a(U, X), \qquad v\nabla_X U = {\nabla^a}_X U,$$
$$h\nabla_X Y = {\nabla^a}_X Y, \qquad v\nabla_X Y = -\frac{1}{2}T^a(X, Y)$$

(h and v are the projections on $\mathcal{H}$ and $\mathcal{V}$ respectively). In particular, $\mathcal{V}$ is a totally geodesic distribution and therefore integrable.

This lemma implies that each point of M has a neighbourhood of the form $M' \times F$, where the fibres $\{p'\} \times F$ are integral manifolds for $\mathcal{V}$. We restrict our considerations to this neighbourhood and denote it again by M. Let $\pi : M \longrightarrow M'$ be the projection.

From Lemma 7.2 we obtain

Lemma 7.3. $(\mathfrak{L}_U g)(X, Y) = 0$ *for $U \in \mathcal{V}$ and $X, Y \in \mathcal{H}$. Hence there exists a Riemannian metric g' on M' such that $\pi : (M, g) \longrightarrow (M', g')$ is a Riemannian submersion.*

From Proposition 3.4 we know that $m = 2n$, $Hol(\nabla^a) \subset \rho(Sp(n)U(1))$ and $T^a = \lambda T_0 + \bar{\lambda}\overline{T}_0$, where $\lambda \neq 0$ is a constant. Thus we have a ∇^a-parallel and compatible with $g_{|\mathcal{H}}$ quaternionic structure $Q = span\{I_{|\mathcal{H}}, J_{|\mathcal{H}}, K_{|\mathcal{H}}\}$ on $\mathcal{H}$, where K and I are defined by (2.3).

Lemma 7.4. *Let $U \in \mathcal{V}$. Then $(h\mathfrak{L}_U(I_{|\mathcal{H}}), h\mathfrak{L}_U(J_{|\mathcal{H}}), h\mathfrak{L}_U(K_{|\mathcal{H}})) = (I_{|\mathcal{H}}, J_{|\mathcal{H}}, K_{|\mathcal{H}}).A(U)$, where*

$$A(U) = \begin{pmatrix} 0 & -2\mathrm{Re}(\lambda e^{2n+1}(U)) & \frac{1}{2}\mathrm{Im}(e^{2n+1}(\nabla^a{}_U e_{2n+1})) \\ 2\mathrm{Re}(\lambda e^{2n+1}(U)) & 0 & -2\mathrm{Im}(\lambda e^{2n+1}(U)) \\ -\frac{1}{2}\mathrm{Im}(e^{2n+1}(\nabla^a{}_U e_{2n+1})) & 2\mathrm{Im}(\lambda e^{2n+1}(U)) & 0 \end{pmatrix}.$$

Hence $Q = span\{I_{|\mathcal{H}}, J_{|\mathcal{H}}, K_{|\mathcal{H}}\}$ projects on a quaternionic structure Q' on M'.

Proof. The first claim follows from the definition of K and I. Thus $h\mathfrak{L}_U(I_{|\mathcal{H}}), h\mathfrak{L}_U(J_{|\mathcal{H}}), h\mathfrak{L}_U(K_{|\mathcal{H}}) \in span\{I_{|\mathcal{H}}, J_{|\mathcal{H}}, K_{|\mathcal{H}}\}$ and therefore Q is projectable. □

Obviously Q' and g' are compatible, i.e., they define an almost quaternionic Hermitian structure on M'.

Lemma 7.5. *∇^a projects on the Levi-Civita connection ∇' of (M', g'). Hence Q' is ∇'-parallel.*

Proof. Since π is a Riemannian submersion, ∇ projects on ∇', i.e., $h\nabla_{X^h}Y^h = (\nabla'_X Y)^h$. Now Lemma 7.2 implies $h\nabla^a{}_{X^h}Y^h = (\nabla'_X Y)^h$, which means that ∇^a also projects on ∇'. Since Q is ∇^a-parallel, the second claim follows from Lemma 7.4. □

Thus (M', g') is quaternionic Kähler if $n > 1$. Now we need to compute its curvature R' to see that it is self-dual and Einstein if $n = 1$ and that the scalar curvature s' is positive for all n. This follows from

Lemma 7.6. $R' = \frac{|T^a|^2}{48n} R_{\mathbb{H}P^n} + R'_{\mathrm{hyper}}$, *where R'_{hyper} is the projection of R_{hyper} (the hyper-Kähler part of R^a) and therefore has the symmetries of a hyper-Kähler curvature tensor.*

Proof. According to O'Neill formulae [21,5]

$$\begin{aligned} R'(X,Y,Z,W) = {} & R(X^h, Y^h, Z^h, W^h) \\ & + g(A_{Y^h}Z^h, A_{X^h}W^h) - g(A_{X^h}Z^h, A_{Y^h}W^h) - 2g(A_{X^h}Y^h, A_{Z^h}W^h), \end{aligned}$$

where

$$A_{X^h}Y^h = \frac{1}{2}v[X^h, Y^h] = \frac{1}{2}v\nabla_{X^h}Y^h.$$

Hence, by Lemma 7.2, $A_{X^h}Y^h = -\frac{1}{2}T^a(X^h, Y^h)$ and from (3.7) we obtain

$$R'(X,Y,Z,W) = R^a(X^h, Y^h, Z^h, W^h) - g(T^a(X^h, Y^h), T^a(Z^h, W^h)).$$

This and Corollary 4.4 yield

$$R'(X,Y,Z,W) = \frac{|T^a|^2}{48n} R^{g_{|\mathcal{H}}}_{\mathbb{H}P^n}(X^h, Y^h, Z^h, W^h) + R_{\mathrm{hyper}}(X^h, Y^h, Z^h, W^h). \quad \square$$

So, up to now we have shown that for all n, (M', g') is quaternionic Kähler with positive scalar curvature $s' = \frac{(n+2)|T^a|^2}{3}$. Let $\mathcal{Z}$ be its twistor space.

For $p \in M$ let $f(p)$ be the projection of $J_{|\mathcal{H}_p}$, i.e., $f(p) = \pi_{*|\mathcal{H}_p}(J_{|\mathcal{H}_p}) \in Q'_{\pi(p)}$. Hence $f(p) \in \mathcal{Z}$ and in this way we obtain a map $f : M \longrightarrow \mathcal{Z}$.

Now it is straightforward (but somewhat lengthy) to show that f gives the desired isomorphism of (M, g, J) and $(\mathcal{Z}, h_{t_1}, J_1)$. The main points are to prove that

1. f maps $g_{|\mathcal{V}}$ on $h_{t_1|\mathcal{V}}$ and $J_{|\mathcal{V}}$ on $J_{1|\mathcal{V}}$, and
2. $f_* X^h = X^h_Z$ (X^h is the horizontal lift on M and X^h_Z is the horizontal lift on $\mathcal{Z}$).

This completes the proof of Theorem 7.1. □

Using Proposition 3.2, Corollaries 3.6 and 5.2 we obtain the obvious counterpart of Theorem 7.1 for nearly Kähler manifolds:

Corollary 7.7. *Let (M, g, J) be a strict nearly Kähler manifold with $Hol(\nabla^a) \subset U(m) \times U(1)$ and $T^a \in \Lambda^{2,0}\mathcal{H}^* \otimes \Lambda^{1,0}\mathcal{V}^* \oplus \Lambda^{0,2}\mathcal{H}^* \otimes \Lambda^{0,1}\mathcal{V}^*$. Then $m = 2n$ and locally (M, g, J) is isomorphic to the twistor space $(\mathcal{Z}, h_{t_1}, J_2)$ of some $4n$-dimensional quaternionic Kähler manifold M' with positive scalar curvature. In particular, $Hol(\nabla^a) \subset \rho_2(Sp(n)U(1))$, with equality if M' is not locally symmetric of rank greater than one.*

Remark. The important point in Corollary 7.7 is that the result is local. For complete manifolds this follows from the results in [20,4] used in the proof of Theorem 6.1.

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